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FRACTIONAL CALCULUS OF GENERALIZED MULTIVARIABLE HURWITZ LERCH ZETA FUNCTION

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ABSTRACT

The history of many members of the astonishingly extensive family of higher transcendental functions can be traced back to such diverse fields as (for example), mathematical physics, analytic number theory, and applied mathematical sciences. These functions are frequently referred to as special functions or mathematical functions. In this article, which is a survey-cum-expository review, our goal is to present a concise introductory overview and survey of some of the recent developments in the theory of several extensively studied higher transcendental functions and their potential applications. In particular, we will focus on the potential applications of these theories. We have decided to give references to a variety of helpful monographs and textbooks on the theory and applications of higher transcendental functions so that people who are interested in investigating this subject can continue their reading and research. Along with their applications, a few fractional calculus operators that are linked with higher transcendental functions have also been taken into consideration here. It is well known that many of the higher transcendental functions, particularly those of the hypergeometric type, which we have investigated in this survey-cum-expository review article, display a kind of symmetry in the sense that they remain invariant when the order of the numerator parameters or when the order of the denominator parameters is arbitrarily changed. This is the case for many of the higher transcendental functions that we have investigated in this article.

Keywords: *gamma; polygamma functions.*

INTRODUCTION

The generalised zeta function, often known as the Hurwitz function $\zeta(s,\nu)$ is defined by

$$\zeta(s,\nu)=\sum_{n=0}^{\infty}\frac{1}{(n+\nu)^{s}}\quad(\Re(s)>1,\ \nu\in\mathbb{C}\setminus\mathbb{Z}_0^-), \qquad \ldots.1$$

It can be thought of as an extension of the Riemann zeta function $\zeta(s):=\zeta(s,1)$. Apostol demonstrated that the analytic continuation formula that is below is correct.

$$\zeta(1-s,\nu)=\frac{2\Gamma(s)}{(2\pi)^s}\sum_{n=1}^{\infty}\frac{\cos(\pi s/2-2\pi\nu n)}{n^s}\quad(0<\nu\leq 1,\ \Re(s)>1), \qquad \text{....2}$$

where r(s) is the well-known Gamma function, for which Euler's integral is defined.

$$\Gamma(s)=\int_0^{\infty}e^{-u}u^{s-1}\,du\quad(\Re(s)>0), \qquad \text{......3}$$

It is also possible to derive it using a transformation formula that is already known for the Lerch zeta function.

$$\phi(x,\nu,s)=\sum_{n=0}^{\infty}\frac{e^{2\pi inx}}{(n+\nu)^s}\quad\left(\Re(s)>1,\ \nu\in\mathbb{C}\setminus\mathbb{Z}_0^-,\ x\in\mathbb{R}\right). \qquad \text{.....4}$$

Take note that it is really simple to understand that $\phi(x,\nu,s)=\zeta(s,\nu)$ when $x\in\mathbb{Z}$. The zeta function developed by Hurwitz and Lerch $\Phi(z,s,\nu)$

$$\Phi(z,s,\nu)=\sum_{n=0}^{\infty}\frac{z^n}{(n+\nu)^s}, \qquad \text{.....5}$$

where $\nu\in\mathbb{C}\setminus\mathbb{Z}_0^-$. The $\Phi(z,s,\nu)$ in (5) converges for all $s\in\mathbb{C}$ when $|z|<1$ and for $\Re(s)>1$ when [z]=1. Here and elsewhere, let $\mathbb{N},\mathbb{Z},\mathbb{R}$, and $\mathbb{C}$ and C be the sets of positive integers, integers, real numbers, and complex numbers, respectively. Furthermore, let us denote $\mathbb{Z}_0^-:=\mathbb{Z}\setminus\mathbb{N}$ and $\mathbb{N}_0:=\mathbb{N}\cup\{0\}$. The several exceptional instances of the Hurwitz–Lerch zeta function (5), which also includes the Riemann zeta function $\zeta(s)$. Both the Hurwitz zeta function (1) and the Lerch zeta function (4) have been the subject of a significant amount of research and practical application. We have decided to use the following examples: In their fifth proposal, Adamchik and Srivastava analysed a set of polygamma functions in terms of the Hurwitz–Lerch zeta function (5). Rassias and Yang [6] investigated several equivalent conditions of a reverse Hilbert-type integral inequality, for which the generalised zeta function was used as an illustration in one of their studies. $\zeta(s,a)$ is demonstrated to have a connection to the most ideal constant factor imaginable. In recent years, a variety of generalisations of the Hurwitz–Lerch zeta function (5) have been the subject of intensive research (see, e.g., [10–19] and the references cited therein). In addition, Choi and Parmar have just very lately presented and researched the following two-variable extension of the Hurwitz–Lerch zeta function (5)

$$\Phi_{a,b,b';c}(x,y,s,\alpha)=\sum_{k,\ell=0}^{\infty}\frac{(a)_{k+\ell}(b)_k(b')_\ell}{(c)_{k+\ell}\,k!\,\ell!}\frac{x^k y^\ell}{(k+\ell+\alpha)^s}, \qquad \text{.....6}$$

where $a,b,b'\in\mathbb{C}$ and $c,\alpha\in\mathbb{C}\setminus\mathbb{Z}_0^-$. The function $\Phi_{a,b,b';c}(x,y,s,\alpha)$ in (6) converges for al $s\in\mathbb{C}$ when $|x|<1$ and $|y|<1$, and for $\Re(s+c-a-b-b')>1$ when $|x|=1$ and $|y|=1$. Here $(\eta)_\nu$ is the Pochhammer symbol give by

$$(\eta)_\nu:=\frac{\Gamma(\eta+\nu)}{\Gamma(\eta)}\quad(\eta+\nu\in\mathbb{C}\setminus\mathbb{Z}_0^-)$$
$$=\begin{cases}1 & (\nu=0),\\ \eta(\eta+1)\cdots(\eta+n-1) & (\nu=n\in\mathbb{N}).\end{cases} \qquad \text{.....7}$$

The following integral formula for the Pochhammer symbol may be derived from equations (3) and (7):

$$(s)_\nu = \frac{1}{\Gamma(s)} \int_0^\infty e^{-u}\, u^{s+\nu-1}\, du \quad (\Re(s+\nu) > 0) \qquad \text{….8}$$

may be acquired with little effort.

In this paper, we aim to establish certain formulas and representations for the extended Hurwitz–Lerch zeta function of two variables (6) in a methodical fashion. These formulas and representations include integral representations, generating functions, derivative formulas, and recurrence relations, among others. In addition, we would like to bring to your attention the fact that the conclusions that have been provided here may be simplified to obtain similar outcomes for a number of Hurwitz–Lerch zeta functions that are less generalised than the extended Hurwitz–Lerch zeta function (6). In addition to this, there are two more generic settings beyond those offered in (6).

Representations of Integral Parts in (6) In the first step of our analysis, we will review a well-known integral form of the extended Hurwitz–Lerch zeta function (6)

$$\Phi_{a,b,b';c}(x,y,s,\alpha) = \frac{1}{\Gamma(s)} \int_0^\infty u^{s-1} e^{-\alpha u} F_1\left[a,b,b';c;xe^{-u},ye^{-u}\right] du, \qquad \text{….9}$$

which approaches the minimum for min $\{\Re(s), \Re(\alpha)\} > 0$ when $|x| \leq 1\ (x \neq 1)$ and $|y| \leq 1\ (y \neq 1)$, and for $\Re(s) > 1$ when x = 1 and y = 1. The Appell hypergeometric function of two variables, denoted by F_1, is defined as follows:

$$\begin{aligned} F_1\left[a,b,b';c;x,y\right] &= \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m(b')_n}{(c)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!} \\ &= \sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{(c)_m}\, {}_2F_1\left[\begin{matrix} a+m,\ b'; \\ c+m; \end{matrix}\ y\right] \frac{x^m}{m!}, \end{aligned} \qquad \text{….10}$$

Where $a, b, b' \in \mathbb{C}, c \in \mathbb{C} \setminus \mathbb{Z}_0^-$ and whose maximum point of convergence is at $\{\Re(x), \Re(y)\} < 1$. Here ${}_pF_q\ (p, q \in \mathbb{N}_0)$ represent the generalised hypergeometric functions; for further information, see also.

It's been brought to my attention that the confluent form of the Appell hyper geometric function F_1 looks like this.

$$\Phi_2[b,b';c;x,y] = \sum_{m,n=0}^{\infty} \frac{(b)_m (b')_n}{(c)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!} \quad (|x| < \infty, |y| < \infty). \qquad \text{…..11}$$

Following is a theorem that asserts additional integral representations of the extended Hurwitz–Lerch zeta function (6). Here, we provide these representations.

Theorem 1. Every single one of the integral representations that follow is valid.

$$\Phi_{a,b,b';c}(x,y,s,\alpha)=\frac{1}{\Gamma(s)\Gamma(b)\Gamma(b')}\int_0^\infty\int_0^\infty\int_0^\infty\Big(u^{s-1}v_1^{b-1}v_2^{b'-1}\exp(-\alpha u-v_1-v_2)\times{}_1F_1\left[a;c;e^{-u}(xv_1+yv_2)\right]\Big)du\,dv_1\,dv_2, \quad \ldots.12$$

where $\min\{\Re(s),\Re(\alpha),\Re(b),\Re(b')\}>0;$

$$\Phi_{a,b,b';c}(x,y,s,\alpha)=\frac{1}{\Gamma(s)\Gamma(a)\Gamma(b)\Gamma(b')}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty\Big(u^{s-1}v_1^{b-1}v_2^{b'-1}v_3^{a-1}\times\exp(-\alpha u-v_1-v_2-v_3)\,{}_0F_1\left[-;c;e^{-u}(xv_1+yv_2)v_3\right]\Big)du\,dv_1\,dv_2\,dv_3, \quad \ldots.13$$

where $\min\{\Re(s),\Re(\alpha),\Re(a),\Re(b),\Re(b')\}>0;$

$$\Phi_{a,b,b';c}(x,y,s,\alpha)=\frac{1}{\Gamma(s)\Gamma(a)}\int_0^\infty\int_0^\infty\Big(u^{s-1}v^{a-1}\exp(-\alpha u-v)\times\Phi_2\left[b,b';c;xe^{-u}v,ye^{-u}v\right]\Big)du\,dv, \quad \ldots.14$$

where $\min\{\Re(s),\Re(\alpha),\Re(a)\}>0$ *and* $\max\{\Re(x),\Re(y)\}<1;$

$$\Phi_{a,b,b';c}(x,y,s,\alpha)=\frac{\Gamma(c)}{\Gamma(s)\Gamma(a)\Gamma(b)\Gamma(b')\Gamma(c-b-b')}\times\int_0^\infty\int_0^\infty\int_0^1\int_0^1\Big(u^{s-1}v^{a-1}\xi^{b-1}\eta^{b'-1}(1-\xi)^{c-b-1} \quad \ldots..15$$

$$\times(1-\eta)^{c-b-b'-1}\exp\left[-\alpha u-v+xe^{-u}v\xi+ye^{-u}v(1-\xi)\eta\right]\Big)d\xi\,d\eta\,du\,dv,$$

where $\min\{\Re(s),\Re(\alpha),\Re(a),\Re(b)\}>0$ *and* $\Re(c-b)>\Re(b')>0;$

$$\Phi_{a,b,b';c}(x,y,s,\alpha)=\frac{\Gamma(\delta)}{\Gamma(s)\Gamma(a)\Gamma(b)\Gamma(b')\Gamma(\delta-b-b')}\int_0^\infty\int_0^\infty\int_0^1\int_0^1\Big(u^{s-1}v^{a-1}\xi^{b-1}\eta^{b'-1}\times(1-\xi)^{\delta-b-1}(1-\eta)^{\delta-b-b'-1}\exp\left[-\alpha u-v+xe^{-u}v\xi+ye^{-u}v(1-\xi)\eta\right]\times{}_1F_1\left[c-\delta;c;-ze^{-u}v\xi-ye^{-u}v(1-\xi)\eta\right]\Big)d\xi\,d\eta\,du\,dv, \quad \ldots\ldots16$$

where $\min\{\Re(s),\Re(\alpha),\Re(a),\Re(b)\}>0$ *and* $\Re(\delta-b)>\Re(b')>0;$

$$\Phi_{a,b,b';c}(x,y,s,\alpha)=\frac{\Gamma(c)}{\Gamma(s)\Gamma(a)\Gamma(\delta)\Gamma(b')\Gamma(c-\delta-b')}\times\int_0^\infty\int_0^\infty\int_0^1\int_0^1\Big(u^{s-1}v^{a-1}\xi^{\delta-1}\eta^{b'-1}(1-\xi)^{c-\delta-1}(1-\eta)^{c-\delta-b'-1}\times\exp\left[-\alpha u-v+xe^{-u}v\xi+ye^{-u}v(1-\xi)\eta\right]{}_1F_1\left[\delta-b;\delta;-xe^{-u}v\xi\right]\Big)d\xi\,d\eta\,du\,dv, \quad 17$$

where $\min\{\Re(s), \Re(\alpha), \Re(a), \Re(\delta)\} > 0$ *and* $\Re(c-\delta) > \Re(b') > 0$;

$$\Phi_{a,b,b';c}(x,y,s,\alpha) = \frac{\Gamma(c)}{\Gamma(s)\Gamma(a)\Gamma(\delta)\Gamma(b')\Gamma(c-\delta-b')} \int_0^\infty\int_0^\infty\int_0^1\int_0^1 \Big(u^{s-1}v^{a-1}\xi^{\delta-1}\eta^{b'-1} \times (1-\xi)^{c-\delta-1}(1-\eta)^{c-\delta-b'-1} \exp\left[-\alpha u - v + xe^{-u}v\xi + ye^{-u}v(1-\xi)\eta\right] \times {}_1F_1\left[b-\delta; c-\delta-b'; xe^{-u}v(1-\xi)(1-\eta)\right] \Big)\, d\xi\, d\eta\, du\, dv, \quad \text{.....18}$$

where $\min\{\Re(s), \Re(\alpha), \Re(a), \Re(\delta)\} > 0$ *and* $\Re(c-\delta) > \Re(b') > 0$;

$$\Phi_{a,b,b';c}(x,y,s,\alpha) = \frac{\Gamma(c)}{\Gamma(s)\Gamma(a)\Gamma(\delta_1)\Gamma(\delta_2)\Gamma(c-\delta_1-\delta_2)} \int_0^\infty\int_0^\infty\int_0^1\int_0^1 \Big(u^{s-1}v^{a-1}\xi^{\delta_1-1}\eta^{\delta_2-1} \times (1-\xi)^{c-\delta_1-1}(1-\eta)^{c-\delta_1-\delta_2-1} \exp\left[-\alpha u - v + xe^{-u}v\xi + ye^{-u}v(1-\xi)\eta\right] \times {}_1F_1\left[\delta_1-b; \delta_1; -xe^{-u}v\xi\right] {}_1F_1\left[\delta_2-b'; \delta_2; -ye^{-u}v(1-\xi)\eta\right] \Big)\, d\xi\, d\eta\, du\, dv, \quad \text{...19}$$

where $\min\{\Re(s), \Re(\alpha), \Re(a), \Re(\delta_1)\} > 0$ *and* $\Re(c-\delta_1) > \Re(\delta_2) > 0$;

$$\Phi_{a,b,b';c}(x,y,s,\alpha) = \frac{\Gamma(c)}{\Gamma(s)\Gamma(a)\Gamma(\delta_1)\Gamma(\delta_2)\Gamma(c-\delta_1-\delta_2)} \times \int_0^\infty\int_0^\infty\int_0^1\int_0^1 \Big(u^{s-1}v^{a-1}\xi^{\delta_1-1}\eta^{\delta_2-1}(1-\xi)^{c-\delta_1-1}(1-\eta)^{c-\delta_1-\delta_2-1} \times \exp\left[-\alpha u - v + xe^{-u}v\xi + ye^{-u}v(1-\xi)\eta\right] \times \Phi_2\left[b-\delta_1; b'-\delta_2; c; xe^{-u}v(1-\xi)(1-\eta), ye^{-u}v(1-\xi)(1-\eta)\right] \Big)\, d\xi\, d\eta\, du\, dv, \quad \text{...20}$$

where $\min\{\Re(s), \Re(\alpha), \Re(a), \Re(\delta_1)\} > 0$ *and* $\Re(c-\delta_1) > \Re(\delta_2) > 0$;

$$\Phi_{a,b,b';c}(x,y,s,\alpha) = \frac{\Gamma(c)}{\Gamma(s)\Gamma(a)\Gamma(b)\Gamma(b')\Gamma(c-b-b')} \times \int_0^\infty\int_0^1\int_0^{1-\eta} \Big(e^{-\alpha u}u^{s-1}\xi^{b-1}\eta^{b'-1}(1-\xi-\eta)^{c-b-b'-1} \times (1-x\xi e^{-u} - y\eta e^{-u})^{-a} \Big)\, d\xi\, d\eta\, du, \quad \text{....21}$$

where $\min\{\Re(s), \Re(\alpha), \Re(b), \Re(b')\} > 0$ *and* $\Re(c-b-b') > 0$;

$$\Phi_{a,b,b';c}(x,y,s,\alpha) = \frac{\Gamma(c)}{\Gamma(s)\Gamma(a)\Gamma(c-a)} \times \int_0^\infty\int_0^1 e^{-\alpha u}u^{s-1}v^{a-1}(1-v)^{c-a-1}(1-vxe^{-u})^{-b}(1-vye^{-u})^{-b'}\, dv\, du, \quad \text{...22}$$

where $\min\{\Re(s), \Re(\alpha), \Re(a)\} > 0$ *and* $\Re(c-a) > 0$;

$$\Phi_{a,b,b';c}(x,y,s,\alpha) = \frac{\Gamma(c)}{\Gamma(s)\Gamma(b)\Gamma(c-b)} \int_0^\infty \int_0^1 \Big(e^{-\alpha u} u^{s-1} v^{c-b-1} (1-v)^{b-1} \times (1-xe^{-u})^{c-a-b} (1-vxe^{-u})^{a-c} {}_2F_1(a,b';c-b;ye^{-u}v) \Big) dv\, du, \quad \ldots 23$$

where $\min\{\Re(s), \Re(\alpha), \Re(b)\} > 0$ *and* $\Re(c-b) > 0$.

Proof. When we apply (8) to the series definition of F1 found in (9), we obtain

$$F_1[a,b,b';c;xe^{-u},ye^{-u}] = \frac{1}{\Gamma(b)\Gamma(b')} \int_0^\infty \int_0^\infty \Big(e^{-v_1-v_2} v_1^{b-1} v_2^{b'-1} \times \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}}{(c)_{m+n}} \frac{(xv_1e^{-u})^m}{m!} \frac{(yv_2e^{-u})^n}{n!} \Big) dv_1\, dv_2. \quad \ldots 24$$

Taking into account the subsequent identity

$$\sum_{m,n=0}^{\infty} f(m+n) \frac{x^m}{m!} \frac{y^n}{n!} = \sum_{N=0}^{\infty} f(N) \frac{(x+y)^N}{N!} \quad \ldots 25$$

We have an integral representation of F1 that may be related to the double series on the right side of equation (24).

$$F_1[a,b,b';c;xe^{-u},ye^{-u}] = \frac{1}{\Gamma(b)\Gamma(b')} \times \int_0^\infty \int_0^\infty e^{-v_1-v_2} v_1^{b-1} v_2^{b'-1} {}_1F_1[a;c;xv_1e^{-u}+yv_2e^{-u}]\, dv_1\, dv_2. \quad \ldots 26$$

In conclusion, by plugging (26) into (9), we arrive at (12).

Similarly, by applying equation (8), we get a form of ${}_1F_1$ that is integral.

$${}_1F_1[a;c;x] = \frac{1}{\Gamma(a)} \int_0^\infty e^{-u} u^{a-1} {}_0F_1[-;c;xu]\, du \quad (\Re(a) > 0). \quad \ldots 27$$

Applying (27) in (12), we obtain (13). Using the following integral formula, which can be found on page 282 of [21], the equation (27),

$$F_1[a,b,b';c;x,y] = \frac{1}{\Gamma(a)} \int_0^\infty u^{a-1} e^{-u} \Phi_2[b,b';c;xu,yu]\, du, \quad \ldots 28$$

where max $\{\Re(x), \Re(y)\} < 1$ and $\Re(a) > 0$, in the integrand of (9) yields (14).

Taking into account the established integral representation of Φ_2 (see [27] Equation (4.2)) in the number 14, we get the following: (15).

In a similar manner, by making use of the previously established integral representations ([27] Equation (4.10)–Equation (4.14) of Φ_2 to (14), accordingly, results in the number (16)– (20).

After applying some well-known integral representations for F1 (see, for example, ([23] p. 76, Equation (1)); see also ([28] Equation (3.2)) and (see, for example, ([23] p. 77, Equation (4)); see also ([28] Equation (3.1)) to (9), respectively, we get (21) and (33). [23] p. 76, Equation (1); see also ([28] Equation ((22).

We derive this result by making use of a well-established integral formulation for F1 (see [28] Equation (3.3)) in (9). (23).

CONCLUSION

In common parlance, higher transcendental functions are also known as special functions or mathematical functions. The history of many members of this astonishingly large family of higher transcendental functions can be traced back to such diverse fields as (for example) analytic number theory, applied mathematical sciences, and mathematical physics. In this article, which is a combination of a survey and an expository review, our goal has been to provide a concise introduction to and survey of some important recent developments in the theory of several extensively studied families of higher transcendental functions (or, more colloquially, special functions) and their potential applications in (for example) mathematical physics, analytic number theory, and applied mathematical sciences. Our survey will be presented in the form of a combination of bullet points and numbered lists. We have decided to give references to a variety of helpful monographs and textbooks on the theory and applications of higher transcendental functions so that people who are interested in investigating this subject can continue their reading and research. We have also looked at a few fractional calculus operators, which are functions related with higher transcendental functions, and provided a quick overview of their applications.

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